

# Comment on “Families of Particles with Different Masses in $\mathcal{PT}$ -Symmetric Quantum Field Theory”

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## Abstract

In a recent letter by Carl M. Bender and S. P. Klevansky an elementary field theory is proposed where the authors claimed that it can account for the existence of family of particles without any employment of group structures. In this work, we relied on the fact that the Dyson-Schwinger equations used by the authors stem from a variational principle and tried to reproduce their results by a mere variational algorithm. We were able to reproduce exactly their results by the employment of two different variational calculations of the vacuum energy in the proposed theory. We showed that the two vacuum solutions obtained by the authors of the commented letter correspond to two non-degenerate vacuum energies except for the  $d = 0$  space-time dimensions where the vacuum states are degenerate. Since any variational calculation of the vacuum energy should be higher than the true (exact) vacuum energy, the most accurate result will correspond to the vacuum that lowers the vacuum energy. We showed that the vacuum with unbroken  $Z_2$  symmetry has lower energy than the vacuum with broken  $Z_2$  symmetry. Accordingly, the theory will prefer to live with the vacuum of unbroken  $Z_2$  symmetry in which the proposed theory is Hermitian. Thus the proposed theory neither has a preferred non-Hermitian representation nor it describes a family of particles as claimed by the authors of the commented work.

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The discovery of possible physical applications of some non-Hermitian theories may lead to a non-problematic description of natural events. The fruitfulness of such trend varies from the discovery of unknown matter phases [1] to the possible solution of great puzzles like the Hierarchy puzzle [2, 3]. Very recently, the authors of Ref. [4] introduced an idea by which they claimed that it can even account for the description of different particle species without the resort to the usual recipe of employing the group theory to the quantum field theory. In this work, we show that their idea does not work for space-time dimensions greater than zero. Even in the zero dimensional case, there exist only one renormalized mass.

In Ref.[4], Carl M. Bender and S. P. Klevansky showed that there exist two vacuum solutions for the massless  $\phi^6$  theory. The two solutions obtained are characterized by two different masses. In their study, they relied on the possible existence of many solutions to the Dyson-Schwinger equations for different boundary conditions on the path of integration in complex field space. First of all, they defined the renormalized masses at different scales. The scale (or subtraction point) once chosen in a specific calculation should be fixed for all other calculations. This criteria is known in the literature by fixing the renormalization scheme [5]. In fact, the formulae given by the authors for the renormalized mass describe many masses (not just two). This is because they defined the renormalized mass at zero vacuum condensate which is described by a fixed scale while the other mass defined at any vacuum condensate which means that this result will give different masses at different scales. In general, a specific vacuum condensate means a specific scale or subtraction point. Accordingly, defining the renormalized mass for unspecified vacuum condensate is meaningless ( see for instance the definition of the renormalized mass at some specific vacuum condensate in Ref.[6]) . The second problem with the commented article is that the Dyson-Schwinger equations stem from a variational principle and although they might have many vacuum solutions, each vacuum will possess a corresponding variational vacuum energy. However, a well known fact is that any variational calculation of the ground state energy is higher than the true (exact) vacuum energy. Accordingly, the most accurate result among the possible vacuum solutions is the one that lowers the vacuum energy. In fact, we used to have many vacuum solutions in quantum field theory which we call them different phases of the theory. However, at some specific scale, the theory will prefer one vacuum solution except at the critical point where degenerate vacua may exist. This trend in understanding the existence of different vacuum solutions in quantum field theory has been asserted experimentally via the measurement of

critical exponents for magnetic systems (for instance) that lie in the same class of universality with a specific quantum field theory ( like Ising model and  $\phi^4$  theory) [7–10].

Rather than the above mentioned known facts about certain terminologies that are clear in literature, in this work, to reinforce our reasoning in a calculational manner, we follow a clear variational calculations that reproduce the same results presented in Ref.[4]. However, the two different variational calculations we use will give two different vacuum energies. As we explained above, the theory will prefer to live with the vacuum that lowers the vacuum energy. In that sense, the theory does have one and only one acceptable vacuum solution and thus describe only one particle and not a family of particles as claimed by the authors of Ref. [4].

To start, consider the Hamiltonian density of the form;

$$H = \frac{1}{2} (\nabla\phi)^2 + \frac{1}{2}\pi^2 + g\phi^6, \quad (1)$$

where the field  $\phi$  has the conjugated momentum field  $\pi$  while  $g$  is a coupling constant.

Let us rewrite this form in terms of a variational mass  $M$  as;

$$\begin{aligned} H &= \frac{1}{2} (\nabla\phi)^2 + \frac{1}{2}\pi^2 + \frac{1}{2}M^2\phi^2 + g\phi^6 - \frac{1}{2}M^2\phi^2, \\ &= H_0 + H_I. \end{aligned} \quad (2)$$

Here  $H_0$  represents the free Hamiltonian  $\frac{1}{2} (\nabla\phi)^2 + \frac{1}{2}\pi^2 + \frac{1}{2}M^2\phi^2$  while  $H_I$  represents the interaction Hamiltonian of the form  $g\phi^6 - \frac{1}{2}M^2\phi^2$ . The vacuum energy of this theory is defined as;

$$E_0(M, g) = \langle 0 | H | 0 \rangle,$$

which up to first order in the coupling has the form;

$$\begin{aligned} E_0(M, g) &= -\frac{1}{2^{d+1}}\pi^{-\frac{d}{2}}M^d\Gamma\left(-\frac{1}{2}d\right) + 15g\Delta^3 - \frac{1}{2}M^2\Delta, \\ \Delta &= \frac{1}{(4\pi)^{\frac{d}{2}}} \frac{\Gamma\left(1 - \frac{d}{2}\right)}{\Gamma(1)} \left(\frac{1}{M^2}\right)^{1-\frac{d}{2}}, \end{aligned} \quad (3)$$

where  $d$  is the dimension of the space-time and  $\Gamma$  is the gamma function. In applying the variational condition of the form  $\frac{\partial E_0(M, g)}{\partial M} = 0$ , we get the result;

$$\left(45h\Delta^2 - \frac{1}{2}M^2\right) \left(-2^{1-d}M^{d-3}\Gamma\left(2 - \frac{1}{2}d\right) \pi^{-\frac{1}{2}d}\right) = 0. \quad (4)$$

This equation gives the result;

$$M_0(g) = \exp \left( \frac{1}{2} \frac{\ln \frac{1}{90g(\Gamma(1-\frac{1}{2}d))^2} + d \ln 4\pi}{d-3} \right), \quad (5)$$

where  $M_0$  is the mass in the unbroken symmetry phase. With this form of the mass, the vacuum energy will depend on the coupling  $g$  as well as some unit mass  $\mu$  that may arise from logarithmic divergences coming from singularities of the gamma function.

Now, let us follow another variational calculation for the vacuum energy. This time we use two variational parameters, the effective field mass  $M$  and the vacuum condensate  $B$ . In this case, the Hamiltonian in Eq.(1) takes the form;

$$H = \frac{1}{2} (\nabla \psi)^2 + \frac{1}{2} \Pi^2 + \frac{1}{2} M^2 \psi^2 + \left( 15gB^4 - \frac{1}{2} M^2 \right) \psi^2 + 20gB^3 \psi^3 + 15gB^2 \psi^4 + 6Bg \psi^5 + g \psi^6 + gB^6, \quad (6)$$

where we used the canonical transformation  $\phi = \psi + B$  and  $\pi = \Pi$ . Here,  $B$  is a constant called the vacuum condensate and  $\Pi = \dot{\psi}$ . Moreover, we dropped out the linear term in the field  $\psi$  since the stability condition will kill it out order by order [11]. Up to first order in the coupling  $g$ , the vacuum energy takes the form [12];

$$E_B(M, B, g) = gB^6 + 15\Delta gB^4 + 45gB^2\Delta^2 - \frac{1}{2} \pi^{-\frac{1}{2}d} \Gamma \left( -\frac{1}{2}d \right) 2^{-d} M^d - M^2 \Delta - 30g\Delta^3. \quad (7)$$

In applying the variational conditions  $\frac{\partial E_B(M, B, g)}{\partial M} = 0$  and  $\frac{\partial E_B(M, B, g)}{\partial B}$ , we get;

$$\begin{aligned} \frac{\partial E_B}{\partial M} &= \frac{-2^{-1-d} \Gamma \left( -\frac{1}{2}d \right) d(-2+d) M^{d-3} \pi^{-\frac{1}{2}d}}{2} (30gB^4 + 180gB^2\Delta - M^2 + 90g\Delta^2) = 0, \\ \frac{\partial E_B}{\partial B} &= B(6B^4 + 60\Delta gB^2 + 90g\Delta^2) = 0. \end{aligned} \quad (8)$$

For  $B = 0$ , we get exactly the same result in Eq.(4). On the other hand, for  $B \neq 0$ , we have the conditions;

$$\begin{aligned} 30gB^4 + 180gB^2\Delta - M^2 + 90g\Delta^2 &= 0, \\ (6B^4 + 60\Delta gB^2 + 90g\Delta^2) &= 0, \end{aligned} \quad (9)$$

where they coincide with the two conditions obtained in Ref.[4]. In solving these two equa-

tions we get;

$$M_B(g) = \exp \left( \frac{1}{2} \frac{\ln \left( -\frac{1}{30gd^2 \left( \Gamma \left( -\frac{1}{2}d \right) \right)^2 (-5+3e^{-d \ln 4} 4^d - \sqrt{10})} \right) + d \ln 4\pi}{-3+d} \right),$$

$$B(g) = \left( \frac{1}{60} 2^d \frac{\pi^{\frac{1}{2}d}}{\Gamma \left( -\frac{1}{2}d \right)} \frac{M_B^{-d+2}}{dg} \left( M_B^2 + \frac{90}{4^d \pi^d} \left( \Gamma \left( -\frac{1}{2}d \right) \right)^2 M_B^{2d-4} d^2 g \right) \right)^{\frac{1}{2}}. \quad (10)$$

where we labeled the mass by the subscript  $B$  to refer to the broken symmetry ( $B \neq 0$ ) case. Since the formulae in Eq.(9) are the same equations obtained in Re.[4], there is no surprise to see that our formulae reproduce the same mass ratios since we have;

$$\frac{M_B}{M_0} = \exp \left( -\frac{1}{2} \frac{2 \ln 2 - \ln 3 + \ln (2 + \sqrt{10})}{-3+d} \right). \quad (11)$$

For  $d = 0, .5, 1, 1.5, 2$  and  $2.5$ , this equation gives  $\frac{M_B}{M_0} = 1.3792, 1.4708, 1.6197, 1.9022, 2.6236$  and  $6.883$  respectively which are exactly the same values presented in Re. [4], however, our calculations accounts for the vacuum energies as well. Using the formula of  $E_0$  in Eq.(3) with the corresponding mass given by Eq.(5) and the formula of  $E_B$  from Eq.(7) with the mass and vacuum condensate are given by Eq.(10), one can get the result;

$$\frac{E_B}{E_0} = \exp \left( -\frac{d}{2} \frac{2 \ln 2 - \ln 3 + \ln (2 + \sqrt{10})}{-3+d} \right), \quad (12)$$

which gives  $\frac{E_B}{E_0} = 1, 1.2128, 1.6197, 2.6236, 6.883$  and  $124.29$  for  $d = 0, .5, 1, 1.5, 2$  and  $2.5$  respectively. This means that the phase with unbroken symmetry and that with broken symmetry are degenerate only at  $d = 0$ . For higher dimensions, the two vacua are non-degenerate and since we followed a direct variational calculations, the most accurate result is that with lower vacuum energy. In other words, the theory will prefer the vacuum with unbroken symmetry ( $B = 0$ ) which means that the theory does describe one and only one particle, not a family of particles as claimed by the authors of Ref. [4]. Also, our result predict that the phase in which the theory is non-Hermitian and  $\mathcal{PT}$ -symmetric is not the preferred phase and thus the theory is Hermitian in the Dirac sense. We need to assert that we followed a clear variational calculations but we get exactly the same results in the commented reference. The reason behind our choice for the variational calculations is to show that the Dyson-Schwinger equations are in fact equivalent to these variational calculations as expected since they stem from variational ansatz.

To conclude, we have shown that the two vacuum solutions obtained in Ref.[4] for the massless  $\phi^6$  theory correspond to two different variational calculations of the vacuum energy. Since any variational calculation of the ground state energy is higher than the true vacuum energy, the theory will prefer to live with the vacuum that lowers the vacuum energy. We found that the vacuum with unbroken  $Z_2$  symmetry is the preferred one as it has the lowest energy among the available vacuum solutions. Since the theory is Hermitian in the preferred phase and non-Hermitian in the other available phase, we conclude that the theory under consideration does not have an acceptable non-Hermitian representation. Moreover, since the vacua are non-degenerate except for the  $d = 0$  case, the theory has only one acceptable vacuum and thus it describes one particle only, not a family of particles as claimed by the authors of Ref. [4].

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